

# Light-cone QCD on the lattice\*

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Ideas and recent results for light-front Hamiltonian quantisation of lattice gauge theories.

## 1. Introduction

A number of physicists have urged the development of a workable light-front (LF) Hamiltonian formulation of QCD [1]. Such a quantisation scheme would yield Lorentz-boost-invariant wavefunctions having a natural constituent structure, providing a power tool in the analysis of hadronic physics. The most obvious applications include inclusive and exclusive hard scattering, and weak and electromagnetic decays, but many new possibilities — yet undreamt of — would be opened up with such a dynamical framework.

So why has nobody done it? The same dynamical reasons that lead to simplified wavefunctions, in particular the infamous triviality of the vacuum state in the presence of high-energy cut-offs, also complicate the construction of renormalised LF Hamiltonians. Non-perturbative effects normally associated with the vacuum must appear explicitly in the Hamiltonian. This limits the scope of a traditional perturbative RG analysis (see, for example, Refs. [2] for attempts to formulate weak-coupling LF RG's). Recently, we have attempted to calculate renormalised LF Hamiltonians in lattice gauge theory, using gauge and Lorentz symmetries of low-energy observables to non-perturbatively fix coupling constant trajectories [3]. So far, this method has been used to study glueballs and the heavy-quark potential in the large- $N$  limit. This talk will review the elements of light-front quantisation of lattice gauge theories, and present some recent results for glueballs.

## 2. Transverse Lattice Hamiltonians.

In  $3 + 1$  spacetime dimensions we introduce a square lattice of spacing  $a$  in the 'transverse' directions  $\mathbf{x} = \{x^1, x^2\}$  and a continuum in the  $\{x^0, x^3\}$  directions. In light-front (LF) coordinates  $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ , we treat  $x^+$  as canonical time and place anti-periodic boundary conditions on  $x^- \sim x^- + \mathcal{L}$ .<sup>2</sup> Both  $1/a$  and  $\mathcal{L}$  are high-energy cut-offs for the LF Hamiltonian  $P^- = (P^0 - P^3)/\sqrt{2}$  that evolves the system in LF time  $x^+$ . This is because the LF free-field dispersion relation for a particle of mass  $\mu$

$$P^- = \frac{\mu^2 + |\mathbf{P}|^2}{2P^+} \quad (1)$$

has LF energy inversely proportional to light-front momentum  $P^+ = (P^0 + P^3)/\sqrt{2}$  (conjugate to  $x^-$ ). Since  $P^+ \geq 0$  and is conserved, the LF vacuum, specified by total  $P^+ = 0$ , can only contain  $P^+ = 0$  modes. The choice of anti-periodic null-plane boundary conditions removes them, but their effects must be recovered in the cut-off Hamiltonian somehow. We propose to do this by constructing general LF Hamiltonians  $P^-$  invariant under gauge transformations and those Lorentz transformations unviolated by the cut-offs, then tuning the remaining couplings to recover the Lorentz symmetries violated by the cut-offs. Lattice gauge formulations are ideal for performing such a procedure non-perturbatively.

Unfortunately, in the usual formulation of lattice gauge theories [5], with degrees of freedom in  $SU(N)$ , it is not straightforward to identify the

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<sup>2</sup>Continuum Lorentz indices are denoted thus  $\alpha, \beta \in \{+, -\}$  and transverse indices thus  $r, s \in \{1, 2\}$ . Repeated indices are summed.

independent degrees of freedom, which is essential for canonical Hamiltonian quantisation. The tricks of ‘equal-time’ Hamiltonian lattice gauge theory in temporal gauge [6] do not carry over to light-front Hamiltonian lattice gauge theory in any convenient way [7]. However, one does not have to choose lattice variables in  $SU(N)$ . It was noted long ago by Bardeen and Pearson [8] that lattice variables  $M_r$  in the space of all complex  $N \times N$  matrices were physically more appropriate for a coarse lattice. In this case, there is no problem in identifying the independent degrees of freedom. The penalty is that one is too far from the continuum to use weak-coupling perturbation theory. But in the case of light-front Hamiltonians, this is of limited use anyway.

The gauge field degrees of freedom below the cut-offs are represented by hermitian gauge potentials  $A_\alpha(\mathbf{x})$  and complex link variables  $M_r(\mathbf{x})$ . On the transverse lattice,  $A_\alpha(\mathbf{x})$  is associated with a site  $\mathbf{x}$ , while  $M_r(\mathbf{x})$  is associated with the link from  $\mathbf{x}$  to  $\mathbf{x} + a\hat{\mathbf{r}}$ . (Each ‘site’  $\mathbf{x}$  is in fact a two-dimensional plane spanned by  $\{x^+, x^-\}$ .) These variables transform under transverse lattice gauge transformations  $U \in SU(N)$  as

$$\begin{aligned} A_\alpha(\mathbf{x}) &\rightarrow U(\mathbf{x})A_\alpha(\mathbf{x})U^\dagger(\mathbf{x}) + i(\partial_\alpha U(\mathbf{x}))U^\dagger(\mathbf{x}) \\ M_r(\mathbf{x}) &\rightarrow U(\mathbf{x})M_r(\mathbf{x})U^\dagger(\mathbf{x} + a\hat{\mathbf{r}}). \end{aligned} \quad (2)$$

Since it will be possible to eliminate  $A_\alpha$  by partial gauge-fixing,  $M_r$  represents the physical transverse polarizations.

The simplest gauge covariant combinations are  $M_r$ ,  $F^{\alpha\beta}$ ,  $\det M$ ,  $\overline{D}^\alpha M$ , *et cetera*, where the covariant derivative is

$$\begin{aligned} \overline{D}_\alpha M_r(\mathbf{x}) &= (\partial_\alpha + iA_\alpha(\mathbf{x}))M_r(\mathbf{x}) \\ &\quad - iM_r(\mathbf{x})A_\alpha(\mathbf{x} + a\hat{\mathbf{r}}). \end{aligned} \quad (3)$$

To proceed, we must make some assumptions about which finite sets of operators to include in a real calculation. The following criteria were used to select operators in  $P^-$  for pure gauge theory:

1. Quadratic LF momentum operator  $P^+$
2. Naive parity restoration as  $\mathcal{L} \rightarrow \infty$ .
3. Transverse (lattice) locality
4. Expansion in gauge-invariant powers of  $M_r$

We do not have space here to explain each of these criteria in detail, but note that the last three can all be straightforwardly checked in principle by systematically relaxing the condition. In all calculations we explicitly extrapolate to the  $\mathcal{L} \rightarrow \infty$  limit<sup>3</sup> at fixed  $a$ , deriving  $P^-$  from a Lagrangian including only dimension 2 operators with respect to  $\{x^+, x^-\}$  coordinates. After fixing to LF gauge  $A_- = 0$  and eliminating the resultant constrained field  $A_+$ , Fock space states will consist of free link-partons  $a^\dagger(k^+)$  obtained by Fourier analysing  $M_r$  in the  $x^-$  coordinate

$$\begin{aligned} M_r &= \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} \left( a_{-r}(k^+) e^{-ik^+ x^-} \right. \\ &\quad \left. + a_r^\dagger(k^+) e^{ik^+ x^-} \right). \end{aligned} \quad (4)$$

For sufficiently large lattice spacing  $a$ ,  $M_r$  remains a massive degree of freedom and there is an energy barrier for the addition of a link-parton to a Fock state. This effect is very pronounced in LF gauge theories [9] and motivates condition 4 above. By expanding  $P^-$  to a given order of  $M_r$  in this regime — the colour-dielectric expansion — we cut off interactions between lower-energy few-parton states and higher-energy many-parton states. The advantage of a cut-off on changes of parton number is that it organizes states into a constituent hierarchy, consistent with energetics. This kind of expansion was suggested in Ref. [10], but only recently have we been able to check that it works in practice.

To leading order, the large- $N$  Lagrangian density satisfying our conditions is

$$\begin{aligned} L_{\mathbf{x}} &= \overline{D}_\alpha M_r(\mathbf{x})(\overline{D}^\alpha M_r(\mathbf{x}))^\dagger \\ &\quad - \frac{1}{2G^2} \text{Tr} \{ F^{\alpha\beta} F^{\alpha\beta} \} - V_{\mathbf{x}} \end{aligned} \quad (5)$$

where

$$\begin{aligned} V_{\mathbf{x}} &= -\frac{\beta}{Na^2} \text{Tr} \{ M_{\text{plaquette}} + M_{\text{plaquette}}^\dagger \} \\ &\quad + \mu^2 \text{Tr} \{ M_r M_r^\dagger \} \\ &\quad + \frac{\lambda_1}{a^2 N} \text{Tr} \{ M_r M_r^\dagger M_r M_r^\dagger \} \\ &\quad + \frac{\lambda_2}{a^2 N} \text{Tr} \{ M_r(\mathbf{x}) M_r(\mathbf{x} + a\hat{\mathbf{r}}) \} \end{aligned}$$

<sup>3</sup>We also used a Tamm-Dancoff cut-off and extrapolated this as well.

$$\begin{aligned}
& M_r^\dagger(\mathbf{x} + a\hat{r})M_r^\dagger(\mathbf{x})\} \\
& + \frac{\lambda_3}{a^2 N^2} (\text{Tr}\{M_r M_r^\dagger\})^2 \\
& + \frac{\lambda_4}{a^2 N} \sum_{\sigma=\pm 2, \sigma'=\pm 1} \text{Tr}\{M_\sigma^\dagger M_\sigma M_{\sigma'}^\dagger M_{\sigma'}\} \\
& + \frac{4\lambda_5}{a^2 N^2} \text{Tr}\{M_1 M_1^\dagger\} \text{Tr}\{M_2 M_2^\dagger\} . \quad (6)
\end{aligned}$$

Collecting everything together, the canonical momenta in LF gauge  $A_- = 0$  become

$$\begin{aligned}
P^+ &= 2 \int dx^- \sum_{\mathbf{x}} \text{Tr}\{\partial_- M_r(\mathbf{x}) \partial_- M_r(\mathbf{x})^\dagger\} \\
P^- &= \int dx^- \sum_{\mathbf{x}} V_{\mathbf{x}} - \text{Tr}\{A_+(\mathbf{x}) J^+(\mathbf{x})\} \\
&\quad - \frac{1}{G^2} \text{Tr}\{\partial_- A_+ \partial_- A_+\} \quad (7) \\
J^+(\mathbf{x}) &= i \left( M_r(\mathbf{x}) \overleftrightarrow{\partial}_- M_r^\dagger(\mathbf{x}) \right. \\
&\quad \left. + M_r^\dagger(\mathbf{x} - a\hat{r}) \overleftrightarrow{\partial}_- M_r(\mathbf{x} - a\hat{r}) \right) \quad (8)
\end{aligned}$$

$A_+$  is a non-dynamical variable in this gauge, and eliminating it introduces non-local interactions thus

$$\begin{aligned}
P^- &= \int dx^- \sum_{\mathbf{x}} \frac{G^2}{4} \text{Tr}\left\{ \frac{J^+}{\partial_-} \frac{J^+}{\partial_-} \right\} \\
&\quad - \frac{G^2}{4N} \text{Tr}\left\{ \frac{J^+}{\partial_-} \right\} \text{Tr}\left\{ \frac{J^+}{\partial_-} \right\} + V_{\mathbf{x}} \quad (9)
\end{aligned}$$

where  $J^+/\partial_- \equiv \partial_-^{-1}(J^+)$ . There is still a residual  $x^-$ -independent gauge invariance generated by the charge  $\int dx^- J^+$ . As originally shown in Refs. [8,10], finite energy states  $|\Psi\rangle$  are subject to the gauge singlet condition  $\int dx^- J^+ |\Psi\rangle = 0$ . In the large- $N$  limit, this means that Fock space at fixed  $x^+$  is formed by connected closed loops of link variables  $M_r$  on the transverse lattice (the  $x^-$  coordinate of each  $M_r$  is unrestricted).

### 3. Glueballs

The dynamical problem is now to diagonalize  $P^-$ , at fixed total momentum  $P^+$ , in the Fock basis (4). To test low-energy eigenfunctions of  $P^-$  (i.e. glueballs) for Lorentz covariance, and so determine the couplings appearing in (6), we also need states at non-zero  $\mathbf{P}$  and methods to

determine the lattice spacing  $a$  and string tension  $\sigma$  [4]. Further details can be found in Refs. [3].

We applied a  $\chi^2$  test for a range of observables that measure violations of Lorentz covariance, including anisotropy of glueball dispersion, asymmetry of the heavy-quark potential, and splitting of Lorentz multiplets in the spectrum. A distinctive one-parameter trajectory  $\mathcal{T}_s$  appears in the space of couplings, largely universal with respect to the precise details of the  $\chi^2$  test, along which  $\chi^2$  is greatly reduced. We believe, as a result, that this is an approximation to a fully Lorentz-covariant scaling trajectory  $\mathcal{T}$  that exists in the infinite-dimensional space of all Hamiltonians and possesses a continuum limit  $a \rightarrow 0$ . The lattice spacing, whose value is deduced as part of the analysis, remains quite large ( $\sim 0.5$  fm) on the piece of  $\mathcal{T}_s$  that we can investigate. Thus,  $\mathcal{T}_s$  may encounter barriers to the continuum and, for this reason, we do not try to extrapolate to  $a = 0$ . Instead, we look for approximate scaling behaviour on coarse lattices and estimate systematic errors empirically from violations of Lorentz covariance.

In fig. 1 we plot the scaling behaviour of the lightest glueball masses  $\mathcal{M}$  along  $\mathcal{T}_s$ . The components are labelled by exact transverse lattice symmetries  $|\mathcal{J}_3|^{\mathcal{P}_1}$  (where  $\mathcal{P}_1 : x^1 \rightarrow -x^1$ ) and grouped into would-be Spin-Parity-Charge-Conjugation multiplets  $\mathcal{J}^{PC}$ . The various components of a Lorentz multiplet become rapidly more covariant, measured by their isotropy and degeneracy, as the link-field mass  $\mu = mG\sqrt{N}$  is reduced.

It is interesting to plot glueball masses in pure gauge theory versus  $1/N^2$ , since this is supposed to be the relevant expansion parameter about  $N = \infty$  [14]. In Figure 2 the  $N = \infty$  data are taken from the best overall  $\chi^2$  of our calculation. There is remarkably little variation of glueball masses with the number of colours. It gives further support to the notion that  $N = 3$  is close to  $N = \infty$  in many situations. In particular, popular flux-tube and string models of the soft gluonic structure of hadrons are typically more appropriate to the large- $N$  limit of QCD (or are independent of  $N$ ). Figure 2 indicates that these models should give worthwhile approximations to  $N = 3$  QCD.

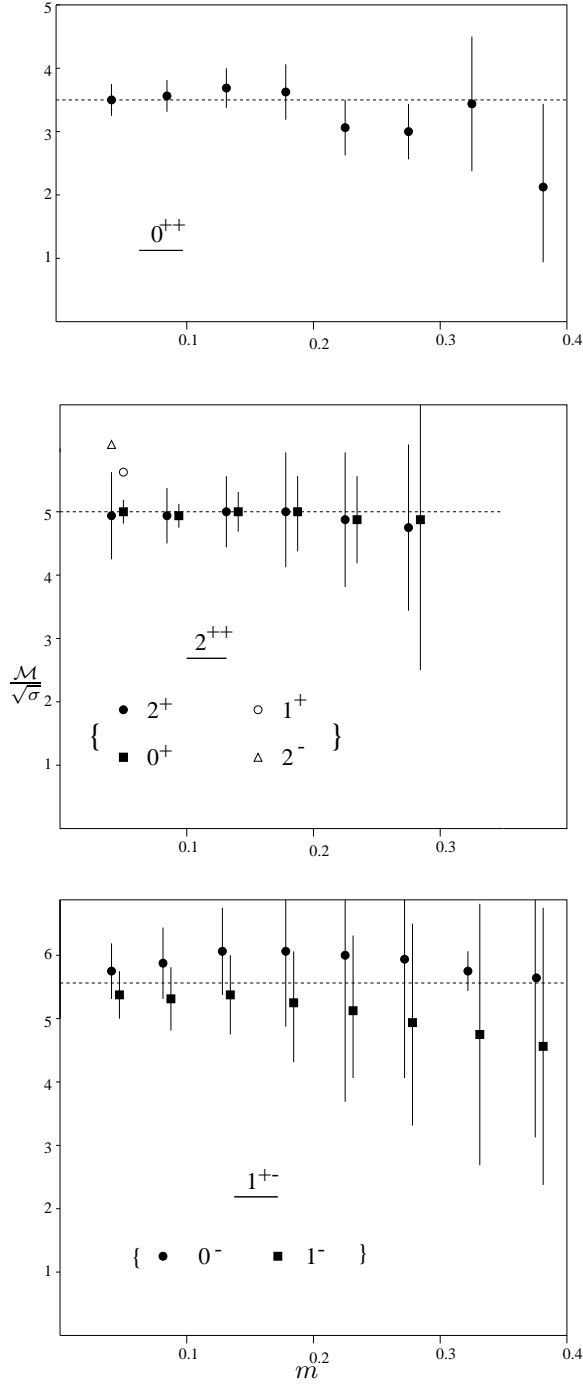


Figure 1. The variation of glueball masses with link-field mass  $m$  along  $\mathcal{T}_s$ . The open-symbol data for the  $2^{++}$  are still too inaccurate for error estimates.

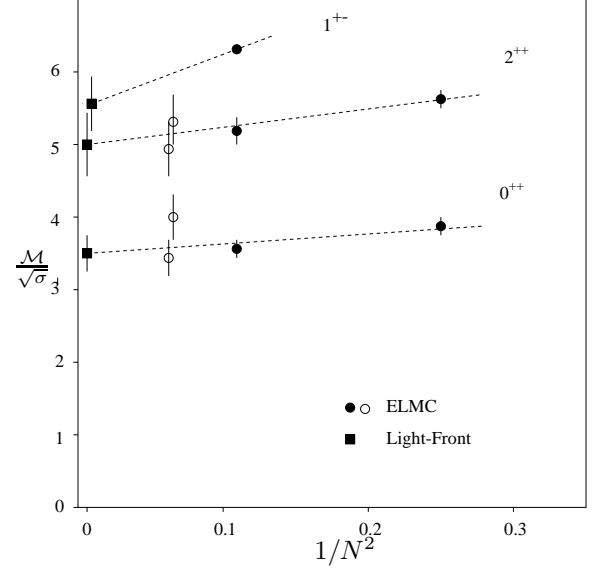


Figure 2. The variation of glueball masses  $\mathcal{M}$  with  $N$  (pure glue). Euclidean Lattice Monte Carlo (ELMC) results are continuum estimates for  $N = 2, 3$  [12,11,13] and fixed lattice spacing results for  $N = 4$  [15]. The dotted lines are to guide the eye, corresponding to leading linear dependence on  $1/N^2$ .

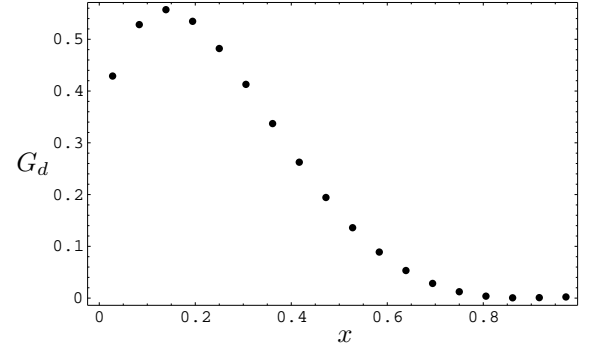


Figure 3. The  $0^{++}$  momentum distribution function at  $a \sim 0.5$  fm.

From the explicit glueball eigenfunctions one can extract various measurements of glueball structure. An interesting quantity is the distribution of longitudinal momentum  $P^+$  among the link partons. In Fig. 3 we plot the quantity

$$G_d(x) = \frac{1}{2\pi x P^+} \int dx^- e^{-ixP^+x^-} \langle \Psi(P^+) | \text{Tr} \{ \partial_- M_r \partial_- M_r^\dagger \} | \Psi(P^+) \rangle ,$$

which measures the probability of finding a link-parton carrying momentum fraction  $x$  of the glueball momentum  $P^+$ . It depends upon the transverse normalisation scale through  $a$ , though our approximation to  $\mathcal{T}$  is too crude to reliably see physical evolution of  $G_d$  with scale.

$G_d$  is related to the gluon distribution; it becomes the gluon distribution in the limit  $a \rightarrow 0$ . Moreover, since  $M_r$  is some collective gluon excitation and the momentum sum rule is satisfied, one would naively expect the gluon distribution at a general scale  $a$  to be softer than  $G_d$ . The  $0^{++}$  glueball does not seem to look like simply a two gluon boundstate, which would have a light-front distribution peaked at  $x = 0.5$ . Once quarks are coupled to the problem, distributions such as Fig. 3 should have distinctive experimental signatures.

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